Simple modules for the restricted Cartan type Lie algebras

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July 13, 2006

Abstract

The simple modules with homogeneous characters are considered, their dimension formulas are determined.

1 Introduction

Let (L,[p]) be a finite-dimensional restricted Lie algebra over an algebraically closed field F, and M an $L-$module. If there exists a linear form $\chi \in L^* = \mathrm{Hom}_F(L, F)$ such that

$$D^p m - D^{[p]} m = \chi(D)^p m$$

for $D \in L$ and $m \in M$. We say that $L-$module M has character χ. From [7], not every module has a character, but at least every simple module has one. If M is an $L-$module with $\chi = 0$, then we call M a *restricted* $L-$module. If $\chi \neq 0$, M is called a *nonrestricted module.*

Let L be the restricted Cartan type algebra over F of characteristic $p \geq 3$. Let $\chi \in L^*$. Let $L = \sum_i L_i$ be the standard grading on L and put $L^i = \sum_{j \geq i} L_j$. The height of the character χ was defined by: $\mathrm{ht}(\chi) = \min\{i \geq -1 | \chi(L^i) = 0\}$.

In 1941, Chang [1] worked with the smallest Witt algebra $W(1,1)$ and determined all the simple modules with arbitrary characters. Later, Strade [6] gave proofs of many of Chang's results in a different approach. Koreshkov [4] studied the next smallest Witt algebra, $W(2,1)$.

Holmes [2] worked with the general Witt algebra $W(n,1)$, gave a uniform treatment of the three cases $\mathrm{ht}(\chi) = -1, 0, 1$ and classified all the simple modules of the restricted Witt algebra $W(n,1)$. He also obtained their dimension formulas. In [3], we classified all the simple modules of the nonexceptional weights with height at most one for the other three types algebras. Namely, special algebras, hamiltonian algebras and contact algebras. Particularly, when the height of the character equals one, all the weights are nonexceptional, in the sense that each simple module is induced by a simple module of its maximal subalgebra.

Then in [9], all the simple modules with the exceptional weights for the type S, H, K are classified. The character with height greater than one was investigated in [10], in which the author proved that all the simple modules with nonsingular or $\Delta -$ invertible characters are induced by the simple modules of their maximal subalgebras.

In the present work, the authors are working with singular homogeneous characters with height greater than one. We show that all simple modules for the Witt algebra with given characters are also induced by the simple modules of their maximal subalgebras. In addition, we determined the dimension of these simple modules.

The paper is organized as follows. In Section 2, we define the restricted Lie algebras of Cartan type. In Section 3, we give theorems about the simplicity for the induced $L-$modules. Then we deduce corollaries particularly for the homogenous characters. As an application, we work out the dimension formula of the simple modules for some homogeneous characters in both Sections 4 and 5.

The research was initiated when the second author was visiting Auburn University. He would like to express his gratitude to the Department of Mathematics of Auburn University for the hospitality.

2 Preliminaries

In this section we describe the simple restricted Lie algebra of Cartan type, drawing most of the notation and results from [7]. Fix $n \in \mathbb{N}$ and let

$a, b \in \mathbb{Z}^n$. We write $a \leqslant b$ if $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant n$ and we write $a < b$ if $a \leqslant b$ but $a \neq b$. If $a, b \geq 0$, define $\binom{a}{b} = \Pi_i \binom{a_i}{b_i}$, where $\binom{a_i}{b_i}$ is the usual binomial coefficient with the convention that $\binom{a_i}{b_i} = 0$ unless $b_i \leqslant a_i$. Set $\mathfrak{C} := \{a \in \mathbb{Z}^n | 0 \leqslant a \leqslant \tau\}$, where $\tau := (p-1, \ldots, p-1)$. The *divided power algebra* $\mathfrak{A} = \mathfrak{A}(n, 1)$ is the associative $F-$ algebra having $F-$ basis $\{x^{(a)} | a \in \mathfrak{C}\}$ and multiplication subject to the rule

$$x^{(a)} x^{(b)} = \begin{cases} \binom{a+b}{a} x^{(a+b)}, & a + b \leqslant \tau \\ 0, & \text{otherwise.} \end{cases}$$

(1) Given $a \in \mathbb{Z}^n$, set $|a| = \sum_i a_i$. Defining $\mathfrak{A}_k = \langle x^{(a)} | a \in \mathfrak{C}, |a| = k\rangle$ and $W_k = \sum_j \mathfrak{A}_{k+1} D_j$, we have the *simple restricted* Witt algebra $W = W(n, 1) = \oplus_{i=-1}^{s_W} W_i$, where $s_W = n(p-1) - 1 = |\tau| - 1$. $W_{-1} = \sum_{i=1}^{n} F D_i$.

(2) Suppose $n \geq 3$, we introduce the mappings

$$D_{ij} : \begin{cases} \mathfrak{A} \to W(n, 1), \\ f \mapsto D_j(f) D_i - D_i(f) D_j. \end{cases}$$

Then the simple *restricted special Lie algebra* is

$$S = S(n, 1) = \langle D_{ij}(f) | f \in \mathfrak{A}, 1 \leqslant i < j \leqslant n\rangle.$$

$S = \oplus_{i=-1}^{s_S} S \cap W_i$ is graded with $s_S = n(p-1) - 2 = |\tau| - 2$. $S_{-1} = W_{-1}$.

(3) Let $r \in \mathbf{N}$ and define

$$\sigma(i) = \begin{cases} 1, & 0 \leqslant i \leqslant r, \\ -1, & r < i \leqslant 2r. \end{cases}$$

For $1 \leqslant i \leqslant 2r$, put $i' = i + \sigma(i) r$.

Define D_H: $\mathfrak{A}(2r, 1) \to W(2r, 1)$ by means of

$$D_H(f) := \sum_{j=1}^{2r} \sigma(j) D_j(f) D_{j'},$$

then by [7], $H' = D_H(\mathfrak{A}(2r, 1))$ is a Lie subalgebra of $W(2r, 1)$. Its subalgebra

$$H = H(2r, 1) = \langle D_H x^{(a)} | 0 \leqslant a < \tau\rangle$$

is called the *simple restricted hamiltonian* Lie algebra. H is a graded subalgebra of W with length $s_H = n(p-1) - 3 = |\tau| - 3$. Directly by the definition, we have $D_H x^{(\epsilon_i)} = \sigma(i) D_{i'}$. Hence we have $H_{-1} = W_{-1}$.

(4) Let $r \in \mathbb{N}$ and put $n = 2r+1$, $\mathfrak{A} = \mathfrak{A}(n,1)$, $W = W(n,1)$. Define a linear mapping D_K: $\mathfrak{A} \to W$ by means of

$$D_K(f) = \sum_{j=1}^{n} f_j D_j,$$

where

$$f_j = x_j D_n(f) + \sigma(j')D_{j'}(f), \quad j \leqslant 2r, f_n = 2f - \sum_{j=1}^{2r} \sigma(j) x_j f_{j'}.$$

Define the Lie product $\langle , \rangle$ on $\mathfrak{A}(2r+1,1)$ by $\langle f,g\rangle := D_K(f)(g) - 2gD_n(f)$. Then $\mathfrak{A}(2r+1,1)$ is a Lie algebra, we denote this Lie algebra by $K'(2r+1,1)$. We define $\|a\| = |a| + a_n + 2$ for $a \in \mathfrak{C}$. The vector spaces $K'(2r+1,1)_i := \langle x^{(a)} | \|a\| = i\rangle$ define a gradation on $K'(2r+1,1)$.

The *simple restricted contact* Lie algebra is then

$$K = K(2r+1,1) = \begin{cases} K'(2r+1,1), & n+3 \not\equiv 0 \mod (p), \\ \oplus_{a<\tau} F x^{(a)}, & n+3 \equiv 0 \mod (p). \end{cases}$$

Then we have $K = \oplus_{i \geq -2} K(2r+1,1)_i$.

In this paper, we denote for the contact algebra K

$$D_1 := x^{(\epsilon_1)}, \ldots, D_{n-1} := x^{(\epsilon_{n-1})}, D_n := 1$$

We let $L_- = L_{-1}$, if $L = W, S, H$, and $L_- = L_{-1} + L_{-2}$ if $L = K$.

We write $\mathrm{Aut}(L)$ for the group of restricted automorphisms of L (Φ: $L \mapsto L$ is restricted provided that $\Phi(D^{[p]}) = \Phi(D)^{[p]}$ for all $D \in L$). Let $\Phi \in \mathrm{Aut}(L)$ and let M^{Φ} be an $L-$module having M as its underlying vector space and $L-$ action given by $x \cdot m = \Phi(x)m$, for $x \in L$ and $m \in M$, where the action on the right is the given one. Then M^{Φ} is simple if and only if M is. Let L be a restricted Lie algebra, and M is an $L-$module with character χ. It is easy to check that M^{Φ} has character χ^{Φ}, where $\chi^{\Phi}(x) = \chi(\Phi(x))$ for $x \in L$. We have $\mathrm{ht}(\chi^{\Phi}) = \mathrm{ht}(\chi)$ by [2, 1.2]. Let $\theta : V \mapsto U$ be a linear transformation of vector spaces over F. If $V = \oplus V_i$ and $U = \oplus U_i$, we say θ is *homogeneous* provided $\theta(V_i) \subseteq U_i$ for each i.

Putting $\mathrm{Aut}^*(L) = \{\Phi \in \mathrm{Aut}(L) | \Phi \text{ is homogeneous }\}$ and $\mathrm{Aut}_1(L) = \{\Phi \in \mathrm{Aut}(L) | (\Phi - 1_L)(L_i) \subseteq L^{i+1} \text{ for each } i\}$, then by [8, Theorem 2], $\mathrm{Aut}(L) = \mathrm{Aut}^*(L) \ltimes \mathrm{Aut}_1(L)$.

Let $\Phi \in \mathrm{Aut}(L)$ and $\chi \in L^*$. Denote $\chi|_{L^0}$ simply by $\chi|$. For any simple $u(L^0, \chi|)$ – module M, denote the induced module $u(L,\chi) \otimes_{u(L^0,\chi|)} M$ by $Z^{\chi}(M)$.

Lemma 1

$$Z^{\chi}(M)^{\Phi} \cong Z^{\chi^{\Phi}}(M^{\Phi})$$

Proof. By the definition, $Z^{\chi^{\Phi}}(M^{\Phi}) = u(L, \chi^{\Phi}) \otimes_{u(L^0, \chi|^{\Phi})} M^{\Phi}$. Since $Z^{\chi}(M)^{\Phi}$ contains M^{Φ} as an L^0 – submodule, by the universal property, there is L – homomorphism f:

$$Z^{\chi^{\Phi}}(M^{\Phi}) \longrightarrow Z^{\chi}(M)^{\Phi},$$

such that $f(\sum D^a \otimes m_a) = \sum D^a \otimes f(m_a)$. Then f is an epimorphism. Since both sides have the same dimension, f is an isomorphism. $\square$

By the Lemma, we have $Z^{\chi}(M) \cong (Z^{\chi^{\Phi}}(M^{\Phi}))^{\Phi^{-1}}$. Then to study $Z^{\chi}(M)$, we may choose a representative in the Aut$W$ – orbit Aut$W \cdot \chi$ such that χ^{Φ} is in a simpler form. We then work on $Z^{\chi^{\Phi}}(M^{\Phi})$. It follows from the lemma that $Z^{\chi^{\Phi}}(M^{\Phi})$ is simple if and only if $Z^{\chi}(M)$ is, and they have the same dimension.

3 Two general theorems

Let $h : A \times B \to F$ be a bilinear form, i.e., h satisfies the following

$$h(a_1 + a_2, b) = h(a_1, b) + h(a_2, b)$$

$$h(a, b_1 + b_2) = h(a, b_1) + h(a, b_2)$$

$$h(ka, b) = h(a, kb) = kh(a, b)$$

for $a, a_i \in A, b, b_i \in B, k \in F$. We denote

$$\mathrm{rad}_L h = \{x \in A | h(a, B) = 0\}, \quad \mathrm{rad}_R h = \{y \in B | h(A, y) = 0\}.$$

Taking a basis of A: $\{u_1, u_2, \ldots, u_m\}$, and that of B: $\{v_1, v_2, \ldots, v_n\}$, let $C_{m\times n} =: (h(u_i, v_j))_{m\times n}$, which is referred to as *the matrix of* h related to the given bases. Denote $r(A)$ the rank of a matrix A. It is easy to see that $\mathfrak{r} =: r(C_{m\times n})$ is invariant with different choice of the basis of A and B. By linear algebra there exist $g \in GL(m)$ and $g' \in GL(n)$ such that

$$gCg' = \begin{pmatrix} I_{\mathfrak{r}} & 0 \\ 0 & 0 \end{pmatrix}$$

Let $(u'_1, u'_2, \ldots, u'_m) = (u_1, u_2, \ldots, u_m)g^T$ and $(v'_1, \ldots, v'_n) = (v_1, \ldots, v_n)g'$, then

$$(h(u'_i, v'_j)) = \begin{pmatrix} I_{\mathfrak{r}} & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $\mathrm{rad}_L h = \langle u'_{\mathfrak{r}+1}, \ldots, u'_m \rangle$ and $\mathrm{rad}_R h = \langle v'_{\mathfrak{r}+1}, \ldots, v'_n \rangle$.

For each $\chi \in L^*$, let $I \subseteq L^{1+\delta_{LK}}$ be an ideal of L^0 satisfying $\chi([I, I]) = 0$. Denote $L^\chi =: \{x \in L^0 | \chi([x, I]) = 0\}$. Obviously $I \subseteq L^\chi$.

We define a skew-symmetric bilinear form B:

$$\underset{(x,y) \longrightarrow \chi([x,y])}{L^0 \times I \longrightarrow F.}$$

Then we have $\mathrm{rad}_L B = L^\chi$. Let $e_1, \ldots, e_t$ be a cobasis of L^χ in L^0. By the discussion above there exists $f_1, \ldots, f_t \in I$, such that the matrix

$$\chi((f_1, \ldots, f_t)^T (e_1, \ldots, e_t))$$

is invertible. In particular, we may choose $\{e_i\}_{i=1}^t$ and $\{f_i\}_{i=1}^t$ such that the matrix is the unit matrix I_t.

Definition. Let L be a Lie algebra and M an $L-$module. P is a subspace of L. If there is $0 \neq v \in M$ such that $x \cdot v = 0$ for all $x \in P$, then v is called an *invariant element* of P in M, or simply a $P-$ *invariant*.

By the definition, if v is a $P-$invariant, then any nonzero multiple of v also is.

In the following, since $u(L_-, \chi) \subseteq u(L, \chi)$ is naturally an L_0- module by Lie product action, we can define the invariants in $u(L_-, \chi)$ for any subspace $P \subseteq L_0$. If each $P-$ invariant in $u(L_-, \chi)$ is in the form $c \cdot 1$, for some $0 \neq c \in F$, we say that P *has only trivial invariants* in $u(L_-, \chi)$.

Lemma 2 *([10, Prop. 2.3]) Let L be a simple restricted Cartan type Lie algebra. Assume $\chi \in L^*$ and $ht(\chi) = h$, $2 \leqslant h \leqslant s_L - \delta_{LK}$. $(i_1, i_2, \ldots, i_n)$ is a rearrangement of the sequence $(1, 2, \ldots, n)$. Fixing $\mathfrak{r}$ with $1 \leqslant \mathfrak{r} \leqslant n$, if there exist elements $g_1, g_2, \ldots, g_{\mathfrak{r}} \in L_k$, for some $k \geq h$, such that the matrix*

$$\chi(\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_r \end{pmatrix} \cdot (D_{i_1}, D_{i_2}, \ldots, D_{i_n}))$$

is in the form $(A_{\mathfrak{r}}|0)$, where $A_{\mathfrak{r}} = (a_{ij})$ is an invertible $\mathfrak{r} \times \mathfrak{r}$ matrix, then every simple $u(L^{h-1}, \chi)-$submodule of $Z^\chi(M)$ is $1-$dimensional. Moreover, *if we denote it by Fv, where $v = \sum_{|s|\leqslant a} D^s \otimes u(s)$, then for all s, $|s| = a$, we have $s_{i_1} = \cdots = s_{i_\mathfrak{r}} = 0$.*

Theorem 1 *Let L be a restricted Lie algebra of Cartan type, and $\chi \in L^*$ with $ht(\chi) = h$, $2 \leqslant h \leqslant s_L - \delta_{LK}$. For I and L^χ given above, assume that $\chi([L^\chi, L^\chi]) = 0$ and assume $[L^\chi, L^\chi]^{[p]} = 0$. Suppose there is a partition of the set $\{1, 2, \ldots, n\}$: $\{1, 2, \ldots, n\} = \mathfrak{I} \cup \mathfrak{J}$, $\mathfrak{I} \cap \mathfrak{J} = \emptyset$. Let $\Gamma_1 = \langle D_i | i \in \mathfrak{I}\rangle$, and $\Gamma_2 = \langle D_i | i \in \mathfrak{J}\rangle$. If there is a subspace $P \subseteq L^\chi \cap L_0$, P has only trivial invariants in $u(\Gamma_1, \chi) \subseteq u(L_-, \chi)$, and if there exist elements $g_1, g_2, \ldots, g_\mathfrak{r} \in L^\chi \cap L_k$, for some $k \geq h$, such that the matrix*

$$\chi(\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_\mathfrak{r} \end{pmatrix} \cdot (D_i, i \in \mathfrak{J} | D_i, i \in \mathfrak{I}))$$

is in the form $(A_\mathfrak{r}|0)$, where $A_\mathfrak{r} = (a_{ij})$ is an invertible $\mathfrak{r} \times \mathfrak{r}$ matrix, then for every simple $u(L^0, \chi)$ − module M, we have

(1) $Z^\chi(M) := u(L, \chi) \otimes_{u(L^0, \chi|)} M$ is a simple $u(L, \chi)$ − module.

(2) $dim Z^\chi(M) = p^{n+t}$, where $t = codim_{L^0}^{L^\chi}$.

Proof. By [7, Coro. 7.5, p.233], for any simple L^χ − module V,

$$\mathrm{Ind}_{L^\chi}^{L^0} =: u(L^0, \chi|) \otimes_{u(L^\chi)} V$$

is a simple $u(L^0, \chi)$ − module. By [7, Lemma 7.2(1), p.230], every simple $u(L^\chi, \chi)$ − module(referred to simply as L^χ − module in the following) is 1 − dimensional. If Fv is a simple L^χ − module, it is clear that there is $\lambda \in \mathrm{Hom}_F(L^\chi, F)$, $x \cdot v = \lambda(x)v$, for all $x \in L^\chi$. Since $x^p \cdot v - x^{[p]} \cdot v = \chi(x)^p v$, we get

$$\lambda^p(x) - \lambda(x) = \chi(x)^p, x \in L^\chi.$$

If $x^{[p]} = 0$, say $x \in L^1$, then we have $\lambda(x) = \chi(x)$. It is clear that each simple L^χ − module is completely determined by λ. We refer to Fv as *the simple L^χ − module with the linear form λ.*

Let $Fv \subseteq M$ be a simple L^χ − submodule with the linear form λ. By [7, Corollary 7.6(1), p.233],

$$M \cong \mathrm{Ind}_{L^\chi}^{L} = u(L^0, \chi|) \otimes_{u(L^\chi)} Fv.$$

Therefore we have

$$Z^\chi(M) \cong u(L, \chi) \otimes_{u(L^\chi)} Fv.$$

Then (2) follows.

We proceed by showing that Fv is the unique simple $L^\chi-$submodule of $Z^\chi(M)$ with linear form λ. Which will imply that $Z^\chi(M)$ is simple.

Recall $\mathfrak{C} =: \{(a_1, \dots, a_n)|0 \leqslant a_i \leqslant p-1, i = 1, \dots, n\}$. For every $a \in \mathfrak{C}$, define $|a| = \sum_{i=1}^n a_i$. Let $e_1, \dots, e_t$ be a cobasis of L^χ in L^0, $f_1, \dots, f_t$ be the elements in I such that $\chi((f_1, \dots, f_t)^T(e_1, \dots, e_t))$ is the unit matrix.

Let M' be a simple $u(L^0, \chi|)-$submodule of $Z^\chi(M)$, and let $Fm \subseteq M'$ be a simple $L^\chi-$submodule with linear form λ. Assume that

$$m = \sum_{\alpha,\beta\in\mathfrak{C},|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta \otimes v,$$

where $D^\alpha =: \Pi_{i=1}^n D_i^{\alpha_i} \in u(L_-, \chi)$, $e^\beta =: \Pi_{i=1}^t e_i^{\beta_i} \in u(L^0, \chi|)$, and $c_{\alpha,\beta} \in F$. Then by Lemma 2, we have $\alpha_i = 0$ for $i \in \mathfrak{J}$ and $|\alpha| = s$.

Applying f_i to m, we have

$$\chi(f_i) \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta \otimes v$$

$$= \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta \otimes f_i v + [f_i, \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta] \otimes v$$
$$= \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta \otimes \chi(f_i) v + [f_i, \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta] \otimes v$$

It follows that

$$(*) \quad [f_i, \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta] \otimes v = 0.$$

Note that $f_i \in I \subseteq L^{1+\delta_{LK}}$. Then $[f_i, D_j] \in L^0$. Since $[\dots[f_i, e_{j_1}]\dots e_{j_s}] \in I$, for any finite sequence $j_1, \dots, j_s$, we have

$$[\dots[f_i, e_{j_1}]\dots e_{j_s}]v = \lambda([\dots[f_i, e_{j_1}]\dots e_{j_s}])v.$$

Also we have $[f_i, e_j]v = \chi([f_i, e_j])v = \delta_{ij} v$. Then using [7, Lemma 7.1, p.229], we have

$$[f_i, \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta] \otimes v$$

$$= \sum_{|\alpha|<s} c_{\alpha,\beta'} D^\alpha e^{\beta'} \otimes v + \sum_{|\alpha|=s,|\beta|=s'} c_{\alpha,\beta} \beta_i D^\alpha e^{\beta-\epsilon_i} \otimes v$$

$$+\sum_{|\alpha|=s,|\beta|<s'-1} c_{\alpha,\beta'} D^\alpha e^{\beta'} \otimes v.$$

Taking the summation of the terms $D^\alpha e^\beta$ on the left $(*)$ with $|\alpha| = s$ and $|\beta| = |s'| - 1$, we have

$$\sum_{|\alpha|=s,|\beta|=s'} c_{\alpha,\beta}\beta_i D^\alpha e^{\beta-\epsilon_i} \otimes v = 0.$$

Then we get $\beta_i = 0$, $i = 1, \dots, t$. It follows that $\beta = 0$ for $|\alpha| = s$.

Taking any $x \in P$, we have

$$\lambda(x)m = \lambda(x) \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta \otimes v = x \cdot m$$

$$= \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta \otimes xv + [x, \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta] \otimes v.$$

Then

$$[x, \sum_{|\alpha|\leqslant s,|\beta|\leqslant s'} c_{\alpha,\beta} D^\alpha e^\beta] \otimes v = 0.$$

Since $x \in L_0$, we have $[x, u(L_-, \chi)_i] \subseteq u(L_-, \chi)_i$, where

$$u(L_-, \chi)_i = \langle D^\alpha \in u(L_-, \chi) || \alpha| = i\rangle.$$

Since $\beta = 0$ for $|\alpha| = s$, we get

$$\sum_{|\alpha|=s} c_{\alpha,0}[x, D^\alpha] \otimes v + \sum_{|\alpha|<s,|\beta|\leqslant t} c'_{\alpha,\beta} D^\alpha e^\beta \otimes v = 0.$$

Thus, $\sum_{|\alpha|=s} c_{\alpha,0}[x, D^\alpha] \otimes v = 0$. Since $v \neq 0$, we have

$$[x, \sum_{|\alpha|=s} c_{\alpha,0} D^\alpha] = 0.$$

Since $\alpha_i = 0$ for all $i \in \mathfrak{J}$ and $|\alpha| = s$, $\sum_{|\alpha|=s} c_{\alpha,0} D^\alpha$ is a $P-$invariant in $u(\Gamma_1, \chi)$. Then we have $s = 0$. This implies that $m = c \otimes v$, for some $c \neq 0$.

Since $M \cong Ind_{L\chi}^{L^0}$, we have $M' = M$. i.e., M is the unique simple $u(L^0, \chi)-$submodule of $Z^\chi(M)$. Then $Z^\chi(M)$ is a simple $u(L, \chi)-$module. $\square$

Note that if $\mathfrak{J} = \{1, 2, \dots, n\}$ and $\mathfrak{J} = \emptyset$, Lemma 2 is then not used in the proof above, so we may allow $\mathrm{ht}(\chi) \leqslant s_L + 1$. Then we get

Theorem 2 *Let L be a restricted Lie algebra of Cartan type, and $\chi \in L^*$. For I and L^χ given above, assume that $\chi([L^\chi, L^\chi]) = 0$ and assume $[L^\chi, L^\chi]^{[p]} = 0$. If there is a subspace $P \subseteq L^\chi \cap L_0$, such that P has only trivial invariants in $u(L_-, \chi)$, then for every simple $u(L^0, \chi) -$ module M, we have*

(1) $Z^\chi(M) := u(L, \chi) \otimes_{u(L^0,\chi)} M$ is a simple $u(L, \chi) -$ module.

(2) $dimZ^\chi(M) = p^{n+t}$, where $t = codim_{L^0}^{L^\chi}$.

Definition. Let L be a restricted Cartan type Lie algebra. If $\chi \in L^*$ satisfies:

(a) $\chi(L_l) \neq 0$ for some $l > 0$;

(b) $\chi(L_i) = 0$, for every $i \geq 0$ and $i \neq l$,

then we say that χ is *homogeneous* with height $l + 1$.

If $\mathrm{ht}(\chi) = 2l + 2$ for some $l > 0$, we take $I = L^{l+1}$. Then it is easy to see that $L^\chi = \sigma \oplus I$, where $\sigma = \{x \in L_0 + \cdots + L_l | \chi([x, I]) = 0\}$.

If χ is homogeneous with height $2l+2$, then restricting the bilinear form B to $L_m \times L_{2l+1-m}$, $m = 0, \dots, l$, we get a bilinear form B_m. Denote

$$L_m^\perp = \mathrm{rad}_L B_m = \{x \in L_m | B_m(x, L_{2l+1-m}) = 0\}.$$

Then it is easy to see that $L_m^\perp = \{x \in L_m | B(x, I) = 0\} = \mathrm{rad}_L B \cap L_m$, and $\sigma = \oplus_{m=0}^l L_m^\perp$. So we have $L^\chi = \oplus_{m=0}^l L_m^\perp \oplus I$. It is easy to check that $\chi([L^\chi, L^\chi]) = 0$. Thus, we have

$$\mathrm{codim}_{L^0}^{L^\chi} = \mathrm{dim} L^0 - \mathrm{dim} L^\chi = \sum_{m=0}^{l} (\mathrm{dim} L_m - \mathrm{dim} L_m^\perp) = \sum_{m=0}^{l} r(C_m),$$

where C_m is the matrix of the bilinear form B_m.

From Theorem 1 we have

Corollary 1 *Let L be a restricted Lie algebra of Cartan type. $\chi \in L^*$ is homogeneous with height $2l+2 \leqslant s_L - \delta_{LK}$. Assume that $\{1, 2, \dots, n\} = \mathfrak{I} \cup \mathfrak{J}$, $\mathfrak{I} \cap \mathfrak{J} = \emptyset$. Let $\Gamma_1 = \langle D_i | i \in \mathfrak{I}\rangle$, and $\Gamma_2 = \langle D_i | i \in \mathfrak{J}\rangle$. Assume $[L_0^\perp, L_0^\perp]^{[p]} = 0$. If there is a subspace $P \subseteq L^\chi \cap L_0$, such that P has only trivial invariants in $u(\Gamma_1, \chi) \subseteq u(L_-, \chi)$, and if there exist elements $g_1, g_2, \dots, g_{\mathfrak{r}} \in L^\chi \cap L_k$, for some $k \geq 2l + 2$, such that the matrix*

$$\chi(\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{\mathfrak{r}} \end{pmatrix} \cdot (D_i, i \in \mathfrak{J} | D_i, i \in \mathfrak{I}))$$

is in the form $(A_{\mathfrak{r}}|0)$, where $A_{\mathfrak{r}} = (a_{ij})$ is an invertible $\mathfrak{r} \times \mathfrak{r}$ matrix, then for any simple $u(L^0, \chi|) -$ module M, $Z^{\chi}(M)$ is a simple $u(L, \chi) -$ module and $dim Z^{\chi}(M) = p^{n+t}$, where $t = codim_{L^0}^{L^{\chi}}$.

From Theorem 2 we have

Corollary 2 *Let L be a restricted Lie algebra of Cartan type. $\chi \in L^*$ is homogeneous with height $2l + 2$. If $[L_0^{\perp}, L_0^{\perp}]^{[p]} = 0$ and $L_0^{\perp}$ has only trivial invariants in $u(L_-, \chi)$, then for any simple $u(L^0, \chi|) -$ module M, $Z^{\chi}(M)$ is a simple $u(L, \chi) -$ module and $dim Z^{\chi}(M) = p^{n+t}$, where $t = codim_{L^0}^{L^{\chi}}$.*

4 Applications of Corollary 2, ht$(\chi) = s + 1$

Let $\mathrm{Aut}^*(W)$ be the group of the homogeneous restricted automorphisms of W. Since $ad^3(tx_iD_j) = 0$ and $p \geq 3$, $exp(adtx_iD_j) \in \mathrm{Aut}^*(W)$. Let E be the subgroup of $\mathrm{Aut}^*(W)$ generated by $\{exp(adtx_iD_j)|t \in F, i \neq j\}$. It is easy to see that a subspace $V \subseteq W$ is a (resp. simple) $W_0 -$ module if and only if it is a (resp. simple) $E -$ module.

Let $s = n(p-1) - 1$. Then s is the largest index i such that $W_i \neq 0$. Using [8, Theorem 2], it is easy to show that $\mathrm{Aut}^*(W)_{|W_s} = \mathrm{GL}(W_s)$ and $E_{|W_s} = \mathrm{SL}(W_s)$. Let $\chi \in W^*$ be homogeneous with height $s + 1$. It is easy to see that $\mathrm{Aut}^*(W) \cdot \chi_{|W_s} = W_s^* - \{0\}$. We assume first that χ is homogeneous and $\chi(x^{(\tau)}D_i) = \delta_{in}$. In addition, we assume that $\chi(D_i) \neq 0$, for some $i < n$.

Using notions defined in Section 3, we have

$$W^{\chi} = \{x \in W^0 | \chi([x, I]) = 0\} = \sum_{i=0}^{r} W_i^{\perp} + I,$$

where

$$I = W^{r+1}, \quad r = \frac{s-1}{2} = \frac{n(p-1)-2}{2}, \quad W_i^{\perp} =: \{x \in W_i | \chi([x, W_{s-i}]) = 0\}.$$

Therefore, W^{χ} is a graded Lie subalgebra of W^0. It is easy to see that $\chi([W^{\chi}, W^{\chi}]) = 0$.

We determine $W_0^{\perp} =: \{x \in W_0 | \chi([x, W_s]) = 0\}$ in the following.

It is easy to check that $x_jD_i \in W_0^{\perp}$ for all $j > i$, and $x_iD_j \in W_0^{\perp}$ for all $i < j \neq n$. Let $x = \sum_{i=1}^{n} a_ix_iD_i + \sum_{i=1}^{n-1} c_ix_iD_n \in W_0^{\perp}$. Then $\chi([x, x^{(\tau)}D_i]) = 0$, for all $i < n$. It follows that $c_i = 0$, for all $i < n$. Then $\chi([\sum_{i=1}^{n} a_ix_iD_i, x^{(\tau)}D_n]) = 0$, which gives $\sum_{i=1}^{n-1} a_i + 2a_n = 0$.

We may choose a set of linearly independent solutions:

$$a_1 = -2a_n, a_2 = \cdots = a_{n-1} = 0; a_2 = -2a_n, a_1 = \cdots = a_{n-1} = 0; \ldots.$$

Therefore $W_0^\perp$ has a $(n-1)-$ dimensional torus

$$T = \langle 2x_1D_1 - x_nD_n, 2x_2D_2 - x_nD_n, \ldots, 2x_{n-1}D_{n-1} - x_nD_n \rangle.$$

It follows that

$$W_0^\perp = \sum_{j>i} Fx_jD_i \oplus T \oplus \sum_{i<j<n} Fx_iD_j.$$

Then we get $\dim W_0^\perp = n^2 - n$.

Lemma 3 *$W_0^\perp$ has only trivial invariants in $u(W_{-1}, \chi)$.*

Proof. Let $m = \sum_{|a| \leqslant s} c_a D^a \in u(W_{-1}, \chi)$ be a $W_0^\perp -$ invariant. Taking $2x_iD_i - x_nD_n \in W_0^\perp$, $i < n$, we have

$$\begin{aligned} 0 &= [(2x_iD_i - x_nD_n), m] \\ &= \sum_{|a| \leqslant s} c_a [2x_iD_i - x_nD_n, D^a] \\ &= -\sum_{|a| \leqslant s} (2a_i - a_n) c_a D^a \end{aligned}$$

This gives us $2a_i = a_n$, $i = 1, \ldots, n-1$.

Taking $i < n$ with $\chi(D_i) \neq 0$, since $x_nD_i \in W_0^\perp$, we have

$$0 = [x_nD_i, m] = \sum_{|a| \leqslant s} (-a_n) c_a D^{a - \epsilon_n + \epsilon_i}$$

Since $0 \neq D^{a-\epsilon_n+\epsilon_i} \in u(W_{-1}, \chi)$ unless $a_n = 0$, we get $a_n = 0$. Thus, $a = 0$ and $m = c_0 \in F$. $\square$

We determine the codimension of W^χ in W^0 next.

Following Section 3, we define the skew-symmetric bilinear form B:

$$\underset{(x,y) \longrightarrow \chi([x,y])}{W^0 \times I \longrightarrow F}.$$

For each $1 \leqslant m \leqslant r = \frac{s-1}{2}$, restricting B to $W_m \times W_{s-m}$, we obtain a bilinear form B_m. Then $W_m^\perp = \mathrm{rad}_L B_m$. Since χ is homogeneous with height $s+1$, by discussions in Section 3,

$$\mathrm{Codim}_{W_0}^{W^\chi} = \sum_{m=0}^{r} \mathrm{Codim}_{W_m}^{W_m^\perp}.$$

For $l \in Z^+$, denote $N_n(l) =: \mathrm{card}\{(a_1, \ldots, a_n) | 0 \leqslant a_i \leqslant p-1, \sum_{i=1}^n a_i = l\}$. We have

$$N_n(l) = \sum_{t=0}^{n} (-1)^t \binom{n}{t} \binom{n+l-tp-1}{n-1}.$$

This is from the last formula on the page

$$http://www.mathpages.com/home/kmath337.htm$$

Then we get $\dim W_l = nN_n(l+1)$.

For $1 \leqslant m \leqslant r$, denote C_m the matrix of B_m related to the standard basis of W_m and W_{s-m}. Then C_m is in the form

$$\begin{pmatrix} \chi(W_m|W_{s-m}) & \underset{i \leqslant N_n(s-m+1), j<n}{x^{(b_i)} D_j} & | & \underset{i \leqslant N_n(s-m+1)}{x^{(b_i)} D_n} \\ --- & --- & | & --- \\ \begin{matrix} x^{(a_1)} D_1 \\ \cdots \\ x^{(a_{N_n(m+1)})} D_{n-1} \end{matrix} & 0 & | & A_1 \\ --- & --- & | & --- \\ \begin{matrix} x^{(a_1)} D_n \\ \cdots \\ x^{(a_{N_n(m+1)})} D_n \end{matrix} & A_2 & | & * \end{pmatrix}.$$

It is easy to check that for every $x^{(a)} D_i \in W_m$, $i < n$,

(1) if $a_i \neq 0$, then $\chi([x^{(a)} D_i, x^{(b)} D_j]) \neq 0$ only if $b = \tau - a + \epsilon_i$ and $j = n$;

(2) if $a_i = 0$, $x^{(a)} D_i \in \mathrm{rad}_L B_m$.

For every $x^{(b)} D_n \in W_{s-m}$, since $s - m > r = \frac{n(p-1)}{2} - 1$, there exists $i < n$ such that $b_i \neq 0$. Hence $x^{(\tau - b + \epsilon_i)} D_i$ is a well defined element of W_m, and $\chi([x^{(\tau-b+\epsilon_i)} D_i, x^{(b)} D_n]) \neq 0$.

Then each row of the matrix A_1 has at most one nonzero entry, and each column of A_1 has at least one nonzero entry. We get $r(A_1) =$ the number of columns of $A_1 = \mathrm{card}\{x^{(b)} D_n | |b| = s - m + 1\} = N_n(s-m+1)$. Also it is easy to see that $r(C_l) = r(A_1) + r(A_2)$.

We now determine $r(A_2)$. For $x^{(a)} D_n \in W_l$, there exists $i < n$ such that $\chi([x^{(a)} D_n, x^{(\tau-a+\epsilon_i)} D_i]) \neq 0$, unless $a_i = 0$, for all $i < n$. The exception occurs only when $m \leqslant p-2$, in which case $\chi([x^{(a)}D_n, x^{(b)}D_i]) = 0$, for all $x^{(b)}D_i \in W_{s-m}$ and $i < n$. Then we have

$$r(A_2) = \mathrm{card}\{a \in \mathfrak{C} | x^{(a)}D_n \in W_m\} - \delta_{l \leqslant p-2} = N_n(m+1) - \delta_{m \leqslant p-2}.$$

Then we have

$$\mathrm{codim}_{W^0}^{W^\chi} = n + \sum_{m=1}^{r} r(C_m)$$

$$= n + \sum_{m=1}^{r} [N_n(m+1) + N_n(s-m+1) - \delta_{m \leqslant p-2}].$$

If $n = 2$, then $[W_0^\perp, W_0^\perp]^{[p]} = 0$. By Corollary 2, for any simple $u(W^0, \chi|)-$ module M, $Z^\chi(M)$ is a simple $u(W, \chi)-$ module with dimension

$$p^{2n + \sum_{m=1}^{r}[N_n(m+1)+N_n(s-m+1)-\delta_{m \leqslant p-2}]}.$$

Theorem 3 *Let $W = W(n,1)$ with $n = 2$, and let $\chi \in W^*$ be homogeneous with height $s+1$. In particular, if there is $\Phi \in Aut^*(W)$, such that $\chi^\Phi(x^{(\tau)}D_i) = \delta_{in}$ and $\chi^\Phi(D_i) \neq 0$ for some $i < n$. Then there are p^{n-1} nonisomorphic simple $u(W, \chi)-$ modules, each of them is induced by a simple module of its maximal subalgebra and has dimension*

$$p^{2n + \sum_{m=1}^{r}[N_n(m+1)+N_n(s-m+1)-\delta_{m \leqslant p-2}]}.$$

Proof. By Lemma 1 we need only to assume that $\chi(x^{(\tau)}D_i) = \delta_{in}$ and $\chi(D_i) \neq 0$ for some $i < n$.

Let Fv be a simple $W^\chi-$ module. Recall that $W_0^\perp$ has a $(n-1)-$ dimensional torus spanned by $h_i = 2x_iD_i - x_nD_n$, $i = 1, \ldots, n-1$. Assume $h_i \cdot v = \lambda_i v$. Since $h_i^p v - h_i^{[p]}v = \chi(h_i)^p v = 0$, $\lambda_i^p = \lambda_i$, or $\lambda_i \in F_p$. Then we get

$$(\lambda_1, \ldots, \lambda_{n-1}) \in F_p^{n-1}.$$

The $(n-1)-$ tuple is referred to as the weight of v. We see that as a $W^\chi-$ module, Fv is completely determined by its weight. Two simple $W^\chi-$modules Fv and Fv' are nonisomorphic if they have different weights. It follows that there are p^{n-1}distinct isomorphism classes of simple $W^\chi-$ *modules*.

Let $Z^\chi(M)$ and $Z^\chi(M')$ be two $u(W, \chi)-$ modules induced by simple $u(W^0, \chi|)-$ modules M and M' respectively. From the proof of Theorem 1 each of them contains a unique simple $W^\chi-$ submodule, denoted Fv and

Fv' respectively, and each is also induced by the $1-$ dimensional simple $W^\chi-$ submodule. Then it follows that $Z^\chi(M) \cong Z^\chi(M')$ if and only if v and v' have the same weights. By Corollary 2 $Z^\chi(M')$ is simple, for any simple $u(W^0, \chi|)-$module M'. Therefore there are at least p^{n-1} isomorphic classes of induced simple $u(W, \chi)-$ modules.

Let $\mathfrak{N}$ be a simple $u(W, \chi)-$ module, $\mathfrak{N}' \subseteq \mathfrak{N}$ be a simple $u(W^0, \chi|)-$ submodule, and let $Fv \subseteq \mathfrak{N}'$ be a simple $W^\chi-$submodule. By [7, Corollary 7.6, p.233], $\mathfrak{N}'$ is induced by Fv. Then the inclusion map $Fv \longrightarrow \mathfrak{N}$ induces a $u(W, \chi)-$ module homomorphism

$$h : Z^\chi(\mathfrak{N}') \cong u(W, \chi) \otimes_{u(W^\chi)} Fv \longrightarrow \mathfrak{N}$$

such that $h(x \otimes v) = x \cdot v$ for every $x \in u(W, \chi)$. Since h is obviously nonzero and both $Z^\chi(\mathfrak{N}')$ and $\mathfrak{N}$ are simple $u(W, \chi)-$modules, h is an isomorphism.

So each simple $u(W, \chi)-$module is isomorphic to some $Z^\chi(M)$. It follows that there are p^{n-1} pairwise nonisomorphic simple $u(W, \chi)-$ modules.

□

Remark: All through this paper, we are only working on the homogeneous characters. But by Lemma 1, if there is $\Phi \in \mathrm{Aut}_1(W)$ and $\chi \in W^*$ such that χ^Φ satisfies Theorem 3 and Theorem 4, then the conclusions of the two theorems also holds for χ, although χ itself need not be homogeneous.

5 Applications of Corollary 1 , ht$(\chi) = 2l+2 < s+1$

Definition. ([10, p.413]) Let $\chi \in L^*$ and $\mathrm{ht}(\chi) = h$, $2 \leqslant h \leqslant s$. We define the *characteristic matrix* of W associated with χ to be $A^\chi := \chi(A)$, the matrix A is given by

$$A := \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_l \end{pmatrix} (D_1, D_2, \dots, D_n),$$

where $\{f_1, \dots, f_l\}$ is a standard basis of W_h. If $r(A^\chi) = n$, χ is referred to as *nonsingular*; if $r(A^\chi) = \mathfrak{r} < n$, then χ is referred to as *singular* with rank $\mathfrak{r}$.

Definition.([10, p.430]) Let $L = W, S, H, K$, and let $\chi \in L^*$ with $\mathrm{ht}(\chi) = h$. For a partition of the set $\{1, \dots, \tilde{n}\}$: $\{1, \dots, \tilde{n}\} = I \cup J$, $I \cap J = \emptyset$, where $I = \{i_1, \dots, i_{\tilde{\mathfrak{r}}}\}$ and $J = \{i_{\tilde{\mathfrak{r}}+1}, \dots, i_{\tilde{n}}\}$, we assume χ satisfies the following:

(a) $\chi((f_1,\ldots,f_l)^T(D_{i_1},\ldots,D_{i_{\tilde{\mathfrak{r}}}}))$ has an invertible $\tilde{\mathfrak{r}}\times\tilde{\mathfrak{r}}$ minor,
(b) $\chi([L_h,D_j])=0$, for every $j\in J$,
(c) there is a L_0- submodule $\triangle\subseteq L_{h-1}$ with $\chi(\triangle)=0$,
(d) there are elements $f_{\tilde{\mathfrak{r}}+1},\ldots,f_{\tilde{n}}\in\triangle$ such that the matrix

$$B=(b_{ij}):=\chi((f_{\tilde{\mathfrak{r}}+1},\ldots,f_{\tilde{n}})^T(D_{i_{\tilde{\mathfrak{r}}+1}},\ldots,D_{i_{\tilde{n}}}))$$

is invertible. Then it is clear that χ is singular with rank $\mathfrak{r}$. χ is called $\triangle$ *−invertible.*

By [10], for every nonsingular or $\Delta-$ invertiable χ, $Z^\chi(M)$ is simple and every simple $u(W,\chi)-$ module is isomorphic to some $Z^\chi(M)$.

For $l>0$, let $2l+2=k(p-1)+r$, $0<r\leqslant p-1$, and let

$$\bar{a}=(p-1,\ldots,\underset{(k)}{p-1},r,0\ldots,0).$$

By [11], $v_1=x^{(\bar{a})}D_n$ is a maximal vector in W_{2l+1}. W_{2l+1} has only two maximal vectors v_1 and v_2. Let $V_1=u(W_0)v_1$ and $V_2=u(W_0)v_2$. If $p\nmid(n+2l+1)$, V_1 and V_2 are the only simple W_0-submodules and $W_{2l+1}=V_1\oplus V_2$.

Let $V_1^*=\{f\in W_{2l+1}^*|f(V_2)=0\}$ and $V_2^*=\{f\in W^*|f(V_1)=0\}$. If $p\nmid(n+2l+1)$, both V_1^* and V_2^* are simple W_0- submodules of W_{2l+1}^* and $W_{2l+1}^*=V_1^*\oplus V_2^*$.

Let $\{x^{(a)}D_i|a\in\mathfrak{C},i=1,\ldots,n\}$ be the standard basis vectors of witt algebra W. For each $x^{(a)}D_i$, we define $C_{x^{(a)}D_i}\in\mathrm{Hom}_F(W_l,F)$ $(0<l\leqslant s)$ by

$$C_{x^{(a)}D_i}(x^{(b)}D_j)=\delta_{ab}\delta_{ij}.$$

In this section, we assume that χ is homogeneous with height $2l+2\neq(n-1)(p-1)$. In particular, we assume that $\chi_{|W_{2l+1}}\in\mathrm{Aut}^*(W)C_{x^{(\bar{a})}D_n}$.

Lemma 4 *([10, Prop. 2.4]) Let Aut(W) be the group of restricted automorphisms of W. Then for every $\Phi\in$ Aut(W), $r(A^\chi)=r(A^{\chi^\Phi})$*

Lemma 5 *If ht$(\chi)=2l+2\geq p-1$, χ is singular with rank $n-k-\delta_{r,p-1}$.*

Proof. Let $\bar{a}=(p-1,\ldots,\underset{k}{p-1},r,0,\ldots,0)$, $0<r\leqslant p-1$. Since $\mathrm{ht}(\chi)\geq p-1$, $|\bar{a}|\geq p-1$. Then we have $k>0$.

By Lemma 4, we may assume that $\chi_{|W_{2l+1}}=C_{x^{(\bar{a})}D_n}$.

If $r<p-1$, then it is clear that $\chi([D_i,x^{(a)}D_j])=0$, for every $i\leqslant k$ and $x^{(a)}D_j\in W_{2l+2}$.

It is also easy to check that

$$-\chi((x^{(\bar{a}+\epsilon_{k+1})}D_{k+1},\ldots,x^{(\bar{a}+\epsilon_n)}D_n)^T(D_{k+1},\ldots,D_n))$$

is the $(n-k)\times(n-k)$ unit matrix, then χ is singular with rank $n-k$.

If $r=p-1$, then $\chi([D_i,x^{(a)}D_j])=0$, for every $i\leqslant k+1$ and $x^{(a)}D_j\in W_{2l+2}$. It is easy to see that

$$-\chi((x^{(\bar{a}+\epsilon_{k+2})}D_{k+2},\ldots,x^{(\bar{a}+\epsilon_n)}D_n)^T(D_{k+2},\ldots,D_n))$$

is the $(n-k-1)\times(n-k-1)$ unit matrix, then χ is singular with rank $n-k-1$. □

Since χ is homogeneous, χ is not $\Delta-$ invertible.

We denote

$$I=W^{l+1},\quad W^{\chi}=\{x\in W^0|\chi([x,I])=0\}.$$

For any simple $u(W^0,\chi|)-$module M, We will show that $Z^{\chi}(M)$ is a simple $u(W,\chi)-$module and compute its dimension for each of the following cases.

5.1 $2l+2>(n-1)(p-1)$

Let $\tilde{A}=:\{(a_1,a_2,\ldots,a_n)|-1\leqslant a_i\leqslant p-1, i=1,\ldots,n\}$. Then we introduce an $\tilde{A}-$ gradation on W (denoted $\mathfrak{G}$) as follows: $\mathfrak{G}(x^{(a)}D_i)=a-\epsilon_i\in\tilde{A}$. $\tilde{A}$ is a completely ordered set with the order $\preccurlyeq$ defined as: $(a_1,\ldots,a_n)\preccurlyeq(b_1,\ldots,b_n)$ iff $a_1=b_1,\ldots,a_{i-1}=b_{i-1},a_i<b_i$ for some $i\geq 1$. Then we have $W_l=\oplus_{\alpha\in\tilde{A}}(W_l)_\alpha$, where $(W_l)_\alpha=\langle x^{(a)}D_i|a-\epsilon_i=\alpha\rangle$.

It is easy to see that for every $i<j$ and $0\neq v\in(W_l)_a$, if $x_iD_j\cdot v\neq 0$, then

$$\mathfrak{G}(x_iD_j\cdot v)=a+\epsilon_i-\epsilon_j\preccurlyeq a=\mathfrak{G}(v).$$

Let $2l+2=(n-1)(p-1)+r$, $0<r<p-1$. Denote $\bar{a}=(p-1,\ldots,p-1,r)\in\tilde{A}$. In this subsection we assume that χ is homogeneous with height $2l+2$. In particular, we assume that $\chi_{|W_{2l+1}}=C_{x^{(\bar{a})}D_n}$. We determine $W_0^{\perp}$ in the following.

For the order $\preccurlyeq$, $\mathfrak{G}(x^{(\bar{a})}D_n)=\bar{a}-\epsilon_n$ is the largest a such that $(W_l)_a\neq 0$. We then have $\sum_{j>i}Fx_jD_i\subseteq W_0^{\perp}$. A similar method as that used in Section 4 applied, we obtain a $(n-1)-$ dimensional torus of $W_0^{\perp}$:

$$T=\{(r-1)x_iD_i+x_nD_n|i=1,\ldots,n-1\}.$$

For each $i < j < n$,there is no $\alpha \in \tilde{A}$ such that $\alpha+\epsilon_i-\epsilon_j = \bar{a}-\epsilon_n$. Therefore $x_iD_j \in W_0^{\perp}$. For each $i < n$, it is easy to check that $x_iD_n \cdot x^{(\bar{a})}D_i = -x^{(\bar{a})}D_n$. Thus, $x_iD_n \notin W_0^{\perp}$. So we have

$$W_0^{\perp} = \sum_{j>i} Fx_jD_i + T + \sum_{i<j<n} Fx_iD_j.$$

Then we get $\dim W_0^{\perp} = n^2 - n$.

We have a partition of the set $\{1,\ldots,n\}$: $\{1,\ldots,n\} = \mathfrak{I} \cup \mathfrak{J}$, where $\mathfrak{I} = \{1,\ldots,n-1\}$ and $\mathfrak{J} = \{n\}$. Taking $x^{(\bar{a}+\epsilon_n)}D_n \in W_{2l+2}$, then we have

$$\chi(x^{(\bar{a}+\epsilon_n)}D_n \cdot (D_1,\ldots,D_{n-1}|D_n)) = (0,\ldots,0|-1).$$

Let $\Gamma_1 = \langle D_i | i = 1,\ldots,n-1\rangle$.

Lemma 6 *$W_0^{\perp}$ has only trivial invariants in $u(\Gamma_1,\chi)$.*

Proof. Let $m = \sum_{|a|\leqslant s, a_n=0} c_aD^a \in u(\Gamma_1,\chi)$ be a $W_0^{\perp}$ − invariant. Taking $(r-1)x_iD_i - x_nD_n \in W_0^{\perp}$, $i < n$, we have

$$0 = [((r-1)x_iD_i - x_nD_n), m]$$
$$= \sum_{|a|\leqslant s} c_a[(r-1)x_iD_i - x_nD_n, D^a]$$
$$= -\sum_{|a|\leqslant s} a_i(r-1)c_aD^a$$

This gives us $a_i = 0$, for all $i < n$. Then we have $m = c \in F$, for some $c \neq 0$. □

For each skew bilinear form $(1 \leqslant m \leqslant l)$

$$B_m : \underset{(x,y)\longrightarrow\chi([x,y])}{W_m \times W_{2l+1-m}} \rightarrow F,$$

denote its matrix by C_m. It is easy to see that

(1) $B_m(x^{(a)}D_s, x^{(b)}D_j) = 0$, if $s, j < n$.

(2) For $x^{(a)}D_s \in W_m (s < n)$, if $a_s = 0$ or $a_n > r$, then $x^{(a)}D_s \in \mathrm{rad}_L B_m$; if $a_s \neq 0$ and $a_n \leqslant r$, there is a unique $b = \bar{a} - a + \epsilon_s$, such that $B_m(x^{(a)}D_s, x^{(b)}D_n) \neq 0$.

(3) For $x^{(a)}D_n \in W_m$, if $a_n > r + 1$, then $x^{(a)}D_n \in \mathrm{rad}_L B_m$.

In calculating $r(C_m)$, we may exclude the elements of $\mathrm{rad}_L B_m$ and $\mathrm{rad}_R B_m$. Then we get a nonzero submatrix of C_m with maximal order, denoted also by C_m.

Using the identity

$$-\binom{\bar{a}}{a-\epsilon_i} = \binom{\bar{a}}{a} \in F, \quad i < n, a_i > 0,$$

we have $C_m =$

$$\left(\begin{array}{cccccc}
\chi(.|.) & x^{(b)}D_1 & \cdots & x^{(b)}D_{n-1} & \underset{b_n\leqslant r}{x^{(b)}D_n} & \underset{b_n=r+1}{x^{(b)}D_n} \\
--- & --- & --- & --- & --- & --- \\
\underset{a_1\neq 0, a_n\leqslant r}{x^{(a)}D_1} & 0 & \cdots & 0 & \underset{\substack{-a+\epsilon_1\\ \binom{\bar a}{a}}}{b=\bar{a}} & 0 \\
--- & --- & --- & --- & --- & --- \\
\cdots\cdots & & & & & \\
--- & --- & --- & --- & --- & --- \\
\underset{a_{n-1}\neq 0, a_n\leqslant r}{x^{(a)}D_{n-1}} & 0 & \cdots & 0 & \underset{\substack{-a+\epsilon_{n-1}\\ \binom{\bar a}{a}}}{b=\bar{a}} & 0 \\
--- & --- & --- & --- & --- & --- \\
\underset{a_n\leqslant r}{x^{(a)}D_n} & \underset{\substack{-a+\epsilon_1\\ \binom{\bar a}{a}}}{b=\bar{a}} & \cdots & \underset{\substack{-a+\epsilon_{n-1}\\ \binom{\bar a}{a}}}{b=\bar{a}} & \underset{\substack{-a+\epsilon_n\\ \binom{\bar a}{a}-\binom{\bar a}{a-\epsilon_n}}}{b=\bar{a}} & \underset{\substack{-a+\epsilon_n\\ \binom{\bar a}{a}}}{b=\bar{a}} \\
--- & --- & --- & --- & --- & --- \\
\underset{a_n=r+1}{x^{(a)}D_n} & 0 & \cdots & 0 & \underset{\substack{-a+\epsilon_n\\ -\binom{\bar a}{a-\epsilon_n}}}{b=\bar{a}} & 0
\end{array}\right).$$

In each block row above, say the row determined by $u = x^{(a)}D_1$ with $a_1 \neq 0$ and $a_n \leqslant r$, we use the notation $\underset{\substack{-a+\epsilon_1\\ \binom{\bar a}{a}}}{b=\bar{a}}$ indicates that when $v = x^{(b)}D_n$ and $b_n \leqslant r$, $\chi(u|v) \neq 0$ only if $b = \bar{a} - a + \epsilon_1$. In particular, $\chi(u|v) = \binom{\bar a}{a}$.

On the second last block row, for each row inside indexed by a with $a_n = 0$, $b = \bar{a} - a + \epsilon_n$ has $b_n = r+1$. So the last nonzero entry is in the last block column. Since there exists some $i < n$, such that $a_i \neq 0$, there is a nonzero entry in the same row, but in the first $n-1$ block columns. By applying elementary column operations, the last block column may be completely eliminated. Similarly the last block row may be eliminated by elementary row operations. Then we have $r(C_m) = r(C'_m)$, where $C'_m =$

$$\begin{pmatrix} \chi(.|.) & x^{(b)}D_1 & \cdots & x^{(b)}D_{n-1} & \underset{b_n\leqslant r}{x^{(b)}D_n} \\ --- & --- & --- & --- & --- \\ \underset{a_1\neq 0, a_n\leqslant r}{x^{(a)}D_1} & 0 & \cdots & 0 & \begin{smallmatrix} b=\bar{a} \\ -a+\epsilon_1 \\ \binom{\bar{a}}{a} \end{smallmatrix} \\ --- & --- & --- & --- & --- \\ \cdots\cdots & & & & \\ --- & --- & --- & --- & --- \\ \underset{a_{n-1}\neq 0, a_n\leqslant r}{x^{(a)}D_{n-1}} & 0 & \cdots & 0 & \begin{smallmatrix} b=\bar{a} \\ -a+\epsilon_{n-1} \\ \binom{\bar{a}}{a} \end{smallmatrix} \\ --- & --- & --- & --- & --- \\ \underset{a_n\leqslant r}{x^{(a)}D_n} & \begin{smallmatrix} b=\bar{a} \\ -a+\epsilon_1 \\ \binom{\bar{a}}{a} \end{smallmatrix} & \cdots & \begin{smallmatrix} b=\bar{a} \\ -a+\epsilon_{n-1} \\ \binom{\bar{a}}{a} \end{smallmatrix} & \begin{smallmatrix} b=\bar{a} \\ -a+\epsilon_n \\ \binom{\bar{a}}{a}-\binom{\bar{a}}{a-\epsilon_n} \end{smallmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \chi(.|.) & \underset{i<n}{x^{(b)}D_i} & \underset{b_n\leqslant r}{x^{(b)}D_n} \\ --- & --- & --- \\ \underset{a_n\leqslant r, a_i\neq 0}{x^{(a)}D_i, i<n} & 0 & A_1 \\ --- & --- & --- \\ \underset{a_n\leqslant r}{x^{(a)}D_n} & A_2 & * \end{pmatrix}.$$

Recall our assumption $2l+2 = (n-1)(p-1)+r$, $1 < r < p-1$. For every $x^{(b)}D_n \in W_{2l+1-m}$ with $b_n \leqslant r$, since

$$|b| = 2l+2-m \geq l+2 = \frac{n-1}{2}(p-1) + \frac{r}{2} + 1 > r,$$

there is $i < n$ such that $b_i \neq 0$. Let $a = \bar{a} - b + \epsilon_i$. Then $a_i \neq 0$, $a_n \leqslant r$ and $\chi([x^{(a)}D_i, x^{(b)}D_n]) \neq 0$.

For each $x^{(a)}D_s$ with $a_s \neq 0$ and $a_n \leqslant r$, there is unique $b = \bar{a} - a + \epsilon_s$, such that $\chi([x^{(a)}D_s, x^{(b)}D_n]) \neq 0$.

So each row of A_1 has only one nonzero entry and each column of A_1 has at least one nonzero entry. It follows that $r(A_1)$ is the number of columns of A_1= $\mathrm{card}\{b \in \mathfrak{C} | b_n \leqslant r, |b| = (2l+2)-m\}$. Also we have $r(C'_m) = r(A_1) + r(A_2)$.

By a similar discussion, we have

$$r(A_2) = \mathrm{card}\{a \in \mathfrak{C} | a_n \leqslant r, |a| = m+1\}.$$

For the convenience, we denote for any $x \in Z^+$

$$f_r(x) := \mathrm{card}\{b \in \mathfrak{C} | b_n \leqslant r, |b| = x + 1\}.$$

It is easy to see that $f_r(x) = \sum_{c=0}^{r} N_{n-1}(x + 1 - c)$. Then we get $r(C_m) = f_r(m) + f_r(2l + 1 - m)$. $m = 1, \ldots, l$.

Thus,

$$\mathrm{codim}_{W^0}^{W^\chi} = \sum_{m=1}^{l} [f_r(m) + f_r(2l + 1 - m)] + n.$$

Let $n = 2$. Then $[W_0^\perp, W_0^\perp]^{[p]} = 0$. Then by Corollary 1, for any simple $u(W^0, \chi|)$ – module M, $Z^\chi(M)$ is a simple $u(W, \chi)$ – module and

$$\mathrm{dim} Z^\chi(M) = p^{\sum_{m=1}^{l}[f_r(m)+f_r(2l+1-m)]+2n}.$$

5.2 $2l + 2 < (n - 1)(p - 1)$

Let $2l + 2 = k(p - 1) + r$, $0 \leqslant r < p - 1$. Following [11], denote $\bar{a} = (\underbrace{p - 1, \ldots, p - 1}_{k}, r, 0, \ldots, 0)$. Then $k + 1 < n$. Assume in this subsection that χ is homogeneous with height $2l + 2$. In particular, we assume that $\chi_{|W_{2l+1}} = C_{x^{(\bar{a})} D_n}$.

Applying a similar method as that used in the last subsection, we have if $r \neq 0$,

$$W_0^\perp = \sum_{j>i} F x_j D_i + T + \sum_{i<j\leqslant k} F x_i D_j + \sum_{k+1<i<j<n} F x_i D_j;$$

if $r = 0$,

$$W_0^\perp = \sum_{j>i} F x_j D_i + T + \sum_{i<j\leqslant k} F x_i D_j + \sum_{k+1\leqslant i<j<n} F x_i D_j,$$

where

$$T = \langle x_i D_i - x_n D_n | i = 1, \ldots, k\rangle \oplus F(x_{k+1} D_{k+1} + r x_n D_n) + \sum_{i>k+1}^{n-1} F x_i D_i.$$

Then we get $\mathrm{dim} T = n - 1$, and $\mathrm{dim} W_0^\perp = t_0 =:$

$$\frac{1}{2}[(n - 1)(n + 2) + k(k - 1) + (n - k - 2)(n - k - 3)] + (n - k - 2)\delta_{r=0}$$

$$= (n - 1)^2 + (k + 1)^2 - kn + \delta_{r=0}(n - k - 2).$$

We have a partition of the set $\{1, \ldots, n\}$: $\{1, \ldots, n\} = \mathfrak{I} \cup \mathfrak{J}$, where $\mathfrak{I} = \{1, \ldots, n-1\}$ and $\mathfrak{J} = \{n\}$. Taking $x^{(\bar{a}+\epsilon_n)} D_n \in W_{2l+2}$, then we have

$$\chi(x^{(\bar{a}+\epsilon_n)} D_n \cdot (D_1, \ldots, D_{n-1} | D_n)) = (0, \ldots, 0| - 1).$$

Let $\Gamma_1 = \langle D_i | i = 1, \ldots, n-1 \rangle$.

Applying a similar argument as that used in the proof of Lemma 6, we have

Lemma 7 *$W_0^{\perp}$ has only trivial invariants in $u(\Gamma_1, \chi)$.*

For $m = 1, \ldots, l$, we define the bilinear form B_m:

$$\underset{(x,y) \longrightarrow \chi([x,y])}{W_m \times W_{2l+1-m} \longrightarrow F}.$$

It is easy to check that for $x^{(a)} D_n \in W_m$, we have

(1) $B_m(x^{(a)} D_i, x^{(b)} D_j) = 0$ if $i < n$ and $j < n$.

(2) If $i \leqslant k$ and $a_i = 0$, then $x^{(a)} D_i \in \mathrm{rad}_L B_m$.

(3) If $a \nleqslant \bar{a}$ and $i < n$, then $x^{(a)} D_i \in \mathrm{rad}_L B_m$.

(4) If $a \leqslant \bar{a}$, and if $i \leqslant k$, $a_i \neq 0$, then there is unique $b = \bar{a} - a + \epsilon_i$, such that $x^{(b)} D_n \in W_{2l+1-m}$ and $B_m(x^{(a)} D_i, x^{(b)} D_n) \neq 0$.

(5) If $a \nleqslant \bar{a} + \epsilon_i$ for some $k < i < n$, $x^{(a)} D_n \in \mathrm{rad}_L B_m$.

(6) If $a \leqslant \bar{a} + \epsilon_i$ for some $k < i < n$, there is unique $b = \bar{a} - a + \epsilon_i$, such that $B_m(x^{(a)} D_n, x^{(b)} D_i) \neq 0$.

Denote the matrix of B_m by C_m, and the maximal nonzero submatrix of C_m

also by C_m. From the discussion above, using the identity

$$-\binom{\bar{a}}{a - \epsilon_i} = \binom{\bar{a}}{a}, \quad a_i > 0, i \leqslant k,$$

we have

$$C_m = \begin{pmatrix} \underset{a_1\neq 0, a\leqslant\bar{a}}{x^{(a)}D_1} \\ ----\\ \dots \\ ----\\ \underset{a_k\neq 0, a\leqslant\bar{a}}{x^{(a)}D_k} \\ ----\\ \underset{a\leqslant\bar{a}}{x^{(a)}D_{k+1}} \\ ----\\ \dots \\ ----\\ \underset{a\leqslant\bar{a}}{x^{(a)}D_{n-1}} \\ ----\\ \underset{a\leqslant\bar{a}}{x^{(a)}D_n} \\ ----\\ \underset{a_{k+1}=r+1, a\leqslant\bar{a}+\epsilon_{k+1}}{x^{(a)}D_n} \\ ----\\ \dots \\ ----\\ \underset{a_{n-1}=1, a\leqslant\bar{a}+\epsilon_{n-1}}{x^{(a)}D_n} \\ ----\\ \underset{a_n=1, a\leqslant\bar{a}+\epsilon_n}{x^{(a)}D_n} \end{pmatrix}$$

$$\cdot(x^{(b)}D_1|\dots|x^{(b)}D_k|x^{(b)}D_{k+1}|\dots|x^{(b)}D_{n-1}|\underset{b\leqslant\bar{a}}{x^{(b)}D_n}|\underset{\substack{b_{k+1}=r+1\\ b\leqslant\bar{a}+\epsilon_{k+1}}}{x^{(b)}D_n}|\dots|\underset{\substack{b_n=1\\ b\leqslant\bar{a}+\epsilon_n}}{x^{(b)}D_n})$$

$$
=
$$

$$
\begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & \substack{b=\bar a\\ -a+\epsilon_1\\ \binom{\bar a}{a}} & 0 & \cdots & 0 \\
 & \cdots & & \cdots & & \cdots & & \cdots & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & \substack{b=\bar a\\ -a+\epsilon_k\\ \binom{\bar a}{a}} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & |*| & *| & \cdots & 0 \\
 & \cdots & & \cdots & & \cdots & & \cdots & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \substack{b=\bar a\\ -a+\epsilon_{n-1}\\ \binom{\bar a}{a}} & 0 \\
\substack{b=\bar a\\ -a+\epsilon_1\\ \binom{\bar a}{a}} & \cdots & \substack{b=\bar a\\ -a+\epsilon_k\\ \binom{\bar a}{a}} & \substack{b=\bar a\\ -a+\epsilon_{k+1}\\ -\binom{\bar a}{a-\epsilon_{k+1}}} & \cdots & 0 & 0 & 0 & \cdots & \substack{b=\bar a\\ -a+\epsilon_n\\ \binom{\bar a}{a}} \\
0 & \cdots & 0 & \substack{b=\bar a\\ -a+\epsilon_{k+1}\\ -\binom{\bar a}{a-\epsilon_{k+1}}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \cdots & & \cdots & & \cdots & & \cdots & & \\
0 & \cdots & 0 & 0 & \cdots & \substack{b=\bar a\\ -a+\epsilon_{n-1}\\ -\binom{\bar a}{a-\epsilon_{n-1}}} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \substack{b=\bar a\\ -a+\epsilon_n\\ -\binom{\bar a}{a-\epsilon_n}} & 0 & \cdots & 0
\end{pmatrix},
$$

where $|*|*| = |\underset{\binom{\bar a}{a}}{b=\bar a - a+\epsilon_{k+1}}|0|$, if $a_{k+1}>0$; if $a_{k+1}=0$, $|*|*| = |0|\underset{\binom{\bar a}{a}}{b=\bar a - a+\epsilon_{k+1}}|$.

For the block row determined by $x^{(\bar a)}D_n's$ with $a \leqslant \bar a$, if $a_n = 0$, then $b = \bar a - a + \epsilon_n$ satisfies $b \leqslant \bar a + \epsilon_n$ and $b_n = 1$. So each row inside has the last nonzero entry in the last block column. Since $m \geq 1$, there is $i \leqslant k+1$ such that $a_i \neq 0$. Let $b = \bar a - a + \epsilon_i$. Then $x^{(b)}D_i \in W_{2l+1-m}$, and $\chi([x^{(a)}D_n, x^{(b)}D_i]) \neq 0$. So at least there is another nonzero entry in the same row but first $k+1$ block columns. Then by applying the elementary column operation, the last block column of C_m can be eliminated.

Similarly, the last block row of C_m can be completely eliminated by elementary row operations. Then C_m has the same rank as the matrix

$$\begin{pmatrix}
0 & \dots & 0 & 0 & \dots & 0 & \substack{b=\bar{a}\\-a+\epsilon_1}\binom{\bar{a}}{a} & 0 & \dots \\
\dots & & \dots & & \dots & & & & \\
0 & \dots & 0 & 0 & \dots & 0 & \substack{b=\bar{a}\\-a+\epsilon_k}\binom{\bar{a}}{a} & 0 & \dots \\
0 & \dots & 0 & 0 & \dots & 0 & |*| & *| & \dots \\
\dots & & \dots & & \dots & & & & \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \substack{b=\bar{a}\\-a+\epsilon_{n-1}}\binom{\bar{a}}{a} \\
\substack{b=\bar{a}\\-a+\epsilon_1}\binom{\bar{a}}{a} & \dots & \substack{b=\bar{a}\\-a+\epsilon_k}\binom{\bar{a}}{a} & \substack{b=\bar{a}\\-a+\epsilon_{k+1}}-\binom{\bar{a}}{a-\epsilon_{k+1}} & \dots & 0 & 0 & 0 & \dots \\
0 & \dots & 0 & \substack{b=\bar{a}\\-a+\epsilon_{k+1}}-\binom{\bar{a}}{a-\epsilon_{k+1}} & \dots & 0 & 0 & 0 & \dots \\
\dots & & \dots & & \dots & & & & \\
0 & \dots & 0 & 0 & \dots & \substack{b=\bar{a}\\-a+\epsilon_{n-1}}-\binom{\bar{a}}{a-\epsilon_{n-1}} & 0 & 0 & \dots
\end{pmatrix}$$

$$= \begin{pmatrix}
\chi(.|.) & \underset{j\leqslant n-1}{x^{(b)}D_j} & \underset{b\leqslant\bar{a}+\epsilon_i,k+1\leqslant i<n}{x^{(b)}D_n} \\
--- & --- & --- \\
\underset{a\leqslant\bar{a},i\leqslant n-1}{x^{(a)}D_i} & 0 & A_1 \\
--- & --- & --- \\
\underset{a\leqslant\bar{a}+\epsilon_i,k+1\leqslant i<n}{x^{(a)}D_n} & A_2 & 0
\end{pmatrix}.$$

Each row(column) of $A_1(A_2)$ has only one nonzero entry.

For each $b \leqslant \bar{a}+\epsilon_i$, $k+1 \leqslant i < n$, if $b_i = (\bar{a})_i + 1$, then let $a = \bar{a} - b + \epsilon_i$. We have $a \leqslant \bar{a}$ and $\chi([x^{(a)}D_i, x^{(b)}D_n]) \neq 0$. If $b \leqslant \bar{a}$, since $m \geq 1$ there is $i \leqslant k+1$ such that $b_i < (\bar{a})_i$. Let $a = \bar{a}-b+\epsilon_i$. Then $\chi([x^{(a)}D_i, x^{(b)}D_n]) \neq 0$.

So each column of A_1 has at least one nonzero entry. Then we have $r(C_m) = r(A_1) + r(A_2)$, and

$$r(A_1) = \text{the number of columns of} \quad A_1$$

$$= \text{card}\{b \in \mathfrak{C} | |b| = 2l+2-m, b \leqslant \bar{a}+\epsilon_i, \quad k+1 \leqslant i < n\}.$$

For any $x \in Z^+$, denote

$$g(x) = \text{card}\{b \in \mathfrak{C} | |b| = x, b \leqslant \bar{a}+\epsilon_i, k+1 \leqslant i < n\}.$$

Then we have

$$g(x) = \text{card}\{b \in \mathfrak{C} \| b| = x, b \leqslant \bar{a}\} + (n-k-1)\text{card}\{b \in \mathfrak{C} \| b| = x-1, b \leqslant \bar{a}\}.$$
$$= \sum_{c=0}^{r} N_k(x-c) + (n-k-1)\sum_{c=0}^{r} N_k(x-1-c).$$

Therefore we get $r(A_1) = g(2l+2-m)$. By a similar discussion as above, we will have $r(A_2) = g(m+1)$. It then follows that

$$r(C_m) = g(m+1) + g(2l+2-m).$$

Thus,

$$\text{codim}_{W_m}^{W_m^{\perp}} = g(m+1) + g(2l+2-m), \quad 1 \leqslant m \leqslant l.$$

Assume $[W_0^{\perp}, W_0^{\perp}]^{[p]} = 0$. For example, in the case $r \neq 0$, let $n = 2, 3$, or let $n = 4$ and $k = 1, 2$. Then by Corollary 1, we have that for any simple $u(W^0, \chi|) -$ module M, $Z^{\chi}(M)$ is a simple $u(W, \chi) -$ module and

$$\text{dim} Z^{\chi}(M) = p^{\sum_{m=1}^{l}(g(m+1)+g(2l+2-m))+n^2-t_0+n}.$$

5.3 The conclusion

Theorem 4 *Let $\chi \in W^*$ be homogeneous with $ht(\chi) = 2l+2 < s+1$ and $2l+2 \neq (n-1)(p-1)$. Assume $[W_0^{\perp}, W_0^{\perp}]^{[p]} = 0$. If there is $\Phi \in Aut^*(W)$, such that $\chi^{\Phi}_{|W_{2l+1}} = C_{x^{(\bar{a})}D_n}$, then there are p^{n-1} pairwise nonisomorphic simple $u(W, \chi) -$ modules. Each of them is induced by the simple submodule of its maximal subalgebra, and has dimension*

$$\begin{cases} p^{\sum_{m=1}^{l}(f_m^r + f_{2l+1-m}^r)+2n}, & \text{if } 2l+2 > (n-1)(p-1), \\ p^{\sum_{m=1}^{l}(g(m+1)+g(2l+2-m)+n^2-t_0+n}, & \text{if } 2l+2 < (n-1)(p-1). \end{cases}$$

Proof. With the results from the two subsections above, then the proof follows by applying a similar argument as that used in the proof of Theorem 3. $\square$

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